Written answers are acceptable so long as they are legible. Remember, you can work with others but you must write the answers on your own. IF YOU WORK WITH OTHERS YOU MUST NOTE WITH WHOM YOU WORKED IN YOUR ANSWER.

Problem 1

We talked about using Bayes rule to modify your probabilities on the basis of new evidence. The idea was that if I gave you some new information today your probabilities today should be your probabilities from yesterday conditioned on the new information. Let P_T be your probability function today and P_Y be your probability function yesterday and let I be the information I gave you. Formally we represent this "principle of conditionalization" by saying:

$$P_T(x) = P_Y(x|I)$$

In this problem I want you to show that if you update in this way it doesn't matter in what order information arrives. Specifically suppose you get two pieces of information (I_1 and I_2) and update on them. You could update in one order:

$$\begin{aligned} P_2(x) &= P(x|I_1) \\ P_3(x) &= P_2(x|I_2) \end{aligned}$$

Or you could update the other way:

$$\begin{aligned} P_2'(x) &= P(x|I_2) \\ P_3'(x) &= P_2'(x|I_1) \end{aligned}$$

Show that regardless of the probability function, $P_3(x) = P_3'(x)$.

Problem 2

The Condorcet cycle is a collection of three people with the following preferences:

$$\begin{aligned} a \succ\ & b \succ c \\ b \succ\ & c \succ a \\ c \succ\ & a \succ b \end{aligned}$$

This preference profile can generate problems for a variety of voting rules. I said in class that this cannot arise with "single peaked preferences" in a single dimension. Let's think about a funny case. Suppose three people live on the edge of a circular lake. None of them own boats. They must collectively decide where to locate a community center and they've chosen three potential points. Each resident wants the community center to be as close as possible in terms of distance around the edge of the lake (because they don't have boats they can't always take the most direct route across the lake).

Can the Condorcet profile arise in this circumstance? If so, give an illustration. If not explain why not.

Problem 3

Which of Arrow's conditions would you prefer to abandon? Why?

Problem 4

Consider the following scenario for a candidate. There is one dimension along which she can locate a position and it will be represented by $[0, 10]$. She must locate her position on a whole number $0, 1, \ldots, 10$. Voters are uniformly distributed over the space, for simplicity assume there is a voter for every real number on the line. Voters will vote for whichever candidate is closest to them on the line. Suppose there are only two candidates and one has already chosen his position, x. Calculate what the optimal position for the other candidate is assuming that she wants to maximize her votes.

Suppose now that you are the first candidate and you know that the second will maximize her expected number of votes. What should you do?

Graduate student problems (extra credit for undergrads)

Problem 5

Suppose a preference relation $\succsim$ that satisfies all of Savage's axioms. We can define the conditional preference relation $\succsim_a$ as is done in Kreps ($x \succsim_a y$ is $x \succsim y$ given a). Prove that $\succsim_a$ satisfies all of Savage's axioms.

Problem 6

There another "hidden assumption" in my characterization of Arrow which I want to bring out with this question. Consider the following set up: There is a finite set of alternatives P (which has at least 3 members). Let R be the set of all preference relations on P (let's use $\succsim$ here) and let S bet the set of all relations on P (not necessarily preference relations). For some arbitrary N, let $f : R^N \to S$.

A few bits of notation. Let $a \succsim_{x_i} b$ mean that in profile $x \in R^N$, individual i prefers a to b. Let $aS_{f(x)}b$ denote the statement f applied to preference profile x results in a relation S such that aSb.

Consider the following conditions on f (these are the standard Arrow conditions):

1. **Unrestricted domain**. $f(x)$ is defined for all $x \in R^N$
2. **No dictator**. There is no i such that $f(x) = x_i$ for all profiles x.
3. **Positive Association** For all $x \in R^N$ and $a, b \in P$, if $aS_{f(x)}b$, then for all profiles x', $aS_{f(x')}b$ so long as x' satisfies these two conditions
   - For all i such that $a \succsim_{x_i} b$, $a \succsim_{x'_i} b$
   - There is some j such that $a \not\succsim_{x_j} b$ but $a \succsim_{x'_j} b$
4. **Citizen Sovereignty** For all $a, b \in P$, there is some x such that $aS_{f(x)}b$
5. **I.I.A.** For all $x \in R^N$ and all $a, b \in P$, if $aS_{f(x)}b$, then for all x', $aS_{f(x')}b$ so long as x' satisfies these two conditions
   - For all i, $a \succsim_{x_i} b$ if and only if $a \succsim_{x'_i} b$
   - For all i, $b \succsim_{x_i} a$ if and only if $b \succsim_{x'_i} a$

Show that there is a social welfare function f that satisfies these conditions. (Try to find one where for all a and b there is some x such that $aS_{f(x)}b$ does not hold.)